Advances in Mathematics **147**, 1–34 (1999)
Article ID aima.1999.1831, available online at http://www.idealibrary.com on IDEAL®

An Euler Relation for Valuations on Polytopes

Daniel A. Klain[1]

School of Mathematics, Georgia Institute of Technology, Atlanta, Georgia 30332-0160
E-mail: klain@math.gatech.edu

Received June 23, 1999

A locally finite point set (such as the set $\mathbb{Z}^n$ of integral points) gives rise to a lattice of polytopes in Euclidean space taking vertices from the given point set. We develop the combinatorial structure of this polytope lattice and derive Euler-type relations for valuations on polytopes using the language of Möbius inversion. In this context a new family of inversion relations is obtained, thereby generalizing

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The essential link between convex geometry and combinatorial theory is the lattice structure of the collection of *polyconvex* sets; that is, the collection of all *finite* unions of compact convex sets in $\mathbb{R}^n$ (also known as the *convex* ring). The present paper is inspired by the role of valuation theory in these two related contexts.

Specifically, we consider the lattice of polytopes (ordered by inclusion) taking all vertices from a prescribed locally finite collection of points in Euclidean space. Much attention has been given to the specific example of *integral polytopes*, the set of polytopes having vertices in the set $\mathbb{Z}^n$ of integer points in $\mathbb{R}^n$. In the more general context we also obtain a locally finite lattice of polytopes in $\mathbb{R}^n$ to which both combinatorial and convex-geometric theories simultaneously apply.

A notion common to both geometric and combinatorial settings is that of *valuation*. Let $\mathscr{C}$ denote a collection of sets closed under finite intersections, and containing the empty set $\varnothing$ as an element. A function $\varphi: \mathscr{C} \to \mathbb{R}$ is called a *valuation* on $\mathscr{C}$ if $\varphi(\varnothing) = 0$, where $\varnothing$ is the empty set, and

$$\varphi(K \cup L) = \varphi(K) + \varphi(L) - \varphi(K \cap L), \tag{1}$$

for all $K, L \in \mathscr{C}$ such that $K \cup L \in \mathscr{C}$ as well.

The notion of valuation serves as both a generalization and a combinatorial analogue of the measures of classical analysis. They include

[1] Research supported in part by NSF Grants DMS-9626688 and DMS-9803571.

0001-8708/99 $30.00
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important functionals, such as volume, surface area, and the Euler characteristic, appearing in such fields as convex geometry, integral geometry, algebraic geometry, and combinatorial lattice theory. A survey of the role of valuations in geometry can be found in [15, 20, 27, 28]. For a combinatorial perspective, see, for example, [17, 22, 32, 33].

The main result of this paper is a general family of Möbius inversion identities that characterize *all* valuations on polytopes. This result, Theorem 4.2, generalizes classical relations of Euler, Dehn–Sommerville, and Macdonald, for functionals on convex polytopes. (See for example [2, 21, 23, 43]). Specifically, the inversion identity of Theorem 4.2 will be seen to generalize the following theorem of Macdonald [23, 39] when applied to convex polytopes.

THEOREM 0.1 (Macdonald's Relation). *Let $\mathscr{T}$ be a finite simplicial complex whose geometric realization is a convex polytope P with boundary ∂P. Denote by $\partial\mathscr{T}$ the subcomplex that triangulates ∂P. Let ψ be a valuation on the lattice of subcomplexes of $\mathscr{T}$. Then*

$$(-1)^{\dim(P)}\,(\psi(\mathscr{T})-\psi(\partial\mathscr{T}))=\sum_{Q\in\mathscr{T}}(-1)^{\dim(Q)}\,\psi(Q) \tag{2}$$

where the sum in (2) *is taken over all simplices Q of the complex $\mathscr{T}$, including the empty simplex $\varnothing$.*

In particular, it will be seen that Theorem 4.2 also implies the classical Dehn–Sommerville equations for convex polytopes.

Some of the results of this article were discovered independently by Ahrens, Gordon, and McMahon [1] and by Edelman and Reiner [9]. The author is grateful to Paul Edelman and Vic Reiner for their helpful comments on an earlier version of this article.

1. LOCALLY FINITE LATTICES OF POLYTOPES

A collection of points $\mathbb{A}\subset\mathbb{R}^n$ is said to be *locally finite* if, for all closed Euclidean balls B in $\mathbb{R}^n$, the set $\mathbb{A}\cap B$ is finite. Throughout this article we denote by $\mathbb{A}$ a given locally finite subset of $\mathbb{R}^n$.

Denote by $\mathscr{I}(\mathbb{A})$ the set of all compact convex polytopes having vertices in $\mathbb{A}$. The set $\mathscr{I}(\mathbb{A})$ is partially ordered by inclusion of sets. Under this partial ordering $\mathscr{I}(\mathbb{A})$ becomes a lattice, in which the meet $P\wedge Q$ of two polytopes P and Q is given by the convex hull of the intersection $P\cap Q\cap\mathbb{A}$, while the join $P\vee Q$ is given by the convex hull of the union $P\cup Q$. Note that $\mathscr{I}(\mathbb{A})$ is not a distributive lattice, nor is it modular. It is, however, a graded, atomic, lower-semimodular lattice. (See [40] for the definitions of these terms.)

The Möbius function of $\mathscr{I}(\mathbb{A})$ is fairly easy to describe. The following proposition is a special case of a theorem of Edelman and Jamison [8].

PROPOSITION 1.1. *Let $P, Q \in \mathscr{I}(\mathbb{A})$ with $\emptyset \subseteq Q \subseteq P$. If $P - Q$ contains exactly k points of $\mathbb{A}$, all of which are vertices of P, then*

$$\mu(Q, P) = (-1)^k,$$

otherwise $\mu(Q, P) = 0$.

In particular, if any positive dimensional face of P contains a point of $\mathbb{A}$ in its relative interior, then $\mu(\emptyset, P) = 0$.

Proof of Proposition 1.1. Let $Q \in \mathscr{I}(\mathbb{A})$, and suppose that P is a convex integral polytope such that $Q \subseteq P$ and $P - Q$ contains exactly k points of $\mathbb{A}$, all of which are vertices of P. In this case the interval $[P, Q]$ in the partial ordering of $\mathscr{I}(\mathbb{A})$ is isomorphic to the Boolean algebra B_k of subsets of a k-element set. This isomorphism is defined by identifying B_k with the subsets of lattice points in $P - Q$.

The Möbius function of B_k is given by $\mu(S, T) = (-1)^{|T| - |S|}$ whenever $S, T \in B_k$ with $S \subseteq T$ (see, for example, [40]). It follows that $\mu(Q, P) = (-1)^k$.

Next, suppose $P - Q$ contains a point $x \in \mathbb{A}$ that is not a vertex of P. In this case, $R \vee (Q \vee \{x\}) = R \vee \{x\} < P$ in $\mathscr{I}(\mathbb{A})$ for all $Q \leqslant R < P$.

Recall from Weisner's Theorem [22, 31, 40] that if L is a finite lattice with Möbius function μ, and if $a \in L$ with $a > 0_L$, then

$$\sum_{x \vee a = 1_L} \mu(0_L, x) = 0. \tag{3}$$

On applying (3) to the interval $[Q, P]$ in $\mathscr{I}$ we obtain

$$\sum_{\substack{R \in [Q, P] \\ R \vee \{x\} = P}} \mu(Q, R) = \sum_{\substack{R \in [Q, P] \\ R \vee (Q \vee \{x\}) = P}} \mu(Q, R) = 0,$$

where $Q \vee \{x\}$ plays the role of a in (3). Since $R \vee \{x\} = P$ only if $R = P$, it follows that $\mu(Q, P) = 0$. ∎

Proposition 1.1 motivates the following definition, also due to Edelman and Jamison [8]. A polytope $P \in \mathscr{I}(\mathbb{A})$ is said to be *free* if $P \cap \mathbb{A}$ consists only of the vertices (extreme points) of P.

We define another (geometric) simplicial complex $\mathrm{Vis}(P)$ of which P is an orthogonal projection. Let $v_1, ..., v_s$ denote all of the points of $P \cap \mathbb{A}$. For each $i = 1, ..., s$, let $y_i \in \mathbb{R}^{n+s}$ be the point

$$y_i = (v_i, e_i), \tag{4}$$

where e_i denotes the ith standard basis vector for $\mathbb{R}^s$.

Define a (geometric) simplicial complex $\mathrm{Vis}(P)$ in $\mathbb{R}^{n+s}$ as follows. Let $y_1, ..., y_s$ be the vertices of $\mathrm{Vis}(P)$. For each distinct pair v_i, v_j such that the line segment $\overline{v_i v_j}$ is *free* in P, let $\overline{y_i y_j}$ be an edge in $\mathrm{Vis}(P)$. Similarly, for $k = 1, ..., s$, let the convex hull of the points $\{y_{i_1}, ..., y_{i_k}\}$ be a $(k-1)$-dimensional simplex in $\mathrm{Vis}(P)$ if and only if the convex hull of the points $\{v_{i_1}, ..., v_{i_k}\}$ is a free polytope (with respect to $\mathbb{A}$) inside P.

Note once again that a distinct pair $v_i, v_j \in P \cap \mathbb{A}$ corresponds to an edge $\overline{y_i y_j}$ in $\mathrm{Vis}(P)$, if and only if $\overline{v_i v_j}$ is free in P; that is, if and only if the points v_i and v_j can "see" each other without any obstructions from points of $\mathbb{A}$ (or "holes" in P) in between. Similarly, the points $\{y_{i_1}, ..., y_{i_k}\}$ span a simplex $\tilde{Q}$ in $\mathrm{Vis}(P)$ if and only if all of the corresponding integral points $\{v_{i_1}, ..., v_{i_k}\}$ can "see" a free convex polytope Q that they span inside P. For this reason we will refer to the simplicial complex $\mathrm{Vis}(P)$ as the *visibility complex* of the polytope P. For example, if P is a free polytope (with respect to $\mathbb{A}$) having $k+1$ vertices, then $\mathrm{Vis}(P)$ is a k-dimensional simplex.

For all positive integers i, define

$$\alpha_i(P) = \text{the number of free polytopes } Q \subseteq P$$
$$\text{such that } Q \text{ has } i \text{ vertices.}$$

In other words, the numbers $\alpha_i(P)$ give the face numbers of the visibility complex $\mathrm{Vis}(P)$ (with indexing shifted forward by one).

Note that the visibility complex of a convex polytope P in $\mathcal{I}(\mathbb{A})$, is not, in general, a topological manifold (with or without boundary), or even a homology manifold. Complications can occur even when P is a convex polygon. In Fig. 1 the set $\mathbb{A}$ is a collection of 9 points in the plane. Let P denote the planar polygon indicated in the left part of the figure. In this example the visibility complex $\mathrm{Vis}(P)$, shown in the right part of Fig. 1, can be expressed as a union of 3 tetrahedra, along with a 2-dimensional triangular flap. In other words, $\mathrm{Vis}(P)$ is not even a pseudomanifold in this elementary example.

For the polygon illustrated in Fig. 1, we have $\alpha_1(P) = 6$, $\alpha_2(P) = 13$, $\alpha_3(P) = 11$, and $\alpha_4(P) = 3$, while $\alpha_k(P) = 0$ for $k \geqslant 5$. Notice that, in this example,

$$\alpha_1(P) - \alpha_2(P) + \alpha_3(P) - \alpha_4(P) = 1.$$

We will see that an analogous relation holds for all convex polytopes.

FIG. 1. A planar polygon and its visibility complex.

Let π: $\mathbb{R}^{n+s} \to \mathbb{R}^n$ be the orthogonal projection of $\mathbb{R}^{n+s}$ onto the first n coordinates. Evidently $\pi(\mathrm{Vis}(P)) = P$. Moreover, the projection π maps the k-dimensional faces of $\mathrm{Vis}(P)$ onto the free polytopes $Q \subseteq P$ having $k+1$ vertices. That is, $\pi(\tilde{Q}) = Q$.

Recall from Proposition 1.1 that for all $P \in \mathscr{I}(\mathbb{A})$ we have

$$\mu(\varnothing, P) = \begin{cases} (-1)^{\alpha_1(P)} & \text{if } P \text{ is free} \\ 0 & \text{otherwise} \end{cases}$$

It follows from the defining properties of the Möbius function [31] that

$$\chi(P) = -\sum_{\varnothing < Q \subseteq P} \mu(\varnothing, Q) = \sum_i (-1)^i \alpha_i(P) = 1 \tag{5}$$

where χ represents both the "Characteristic" of the lattice $\mathscr{I}(\mathbb{A})$ (see, for example, [31, 32]) and the topological Euler characteristic of a convex polytope P [29].

An even more general construction will also be of use in the sections following. For $P \in \mathscr{I}(\mathbb{A})$, denote by $\Delta(P)$ the abstract simplex whose vertices are in one-to-one correspondence with the points of $P \cap \mathbb{A}$. In other words, $\Delta(P)$ can be realized geometrically as the simplex whose vertices are the affinely independent points $y_1, ..., y_s$ of (4); namely, the convex hull of $\mathrm{Vis}(P)$ in $\mathbb{R}^{n+s}$. Evidently the visibility complex $\mathrm{Vis}(P)$ is a subcomplex of $\Delta(P)$, with $\mathrm{Vis}(P) = \Delta(P)$ if and only if P is *free* with respect to $\mathbb{A}$. Once again the projection map π: $\mathbb{R}^{n+s} \to \mathbb{R}^n$ maps $\Delta(P)$ onto P.

2. ORDER IDEALS AND INDUCED VALUATIONS

Let $\mathscr{B}(\mathbb{A})$ denote the lattice of all *finite* sets of polytopes in $\mathscr{I}(\mathbb{A})$; that is, the collection of finite subsets of $\mathscr{I}(\mathbb{A})$. The set $\mathscr{B}(\mathbb{A})$ is a locally finite distributive lattice, with meet and join given respectively by intersection and union of finite subsets of $\mathscr{I}(\mathbb{A})$.

A subset $A \subseteq \mathscr{I}(\mathbb{A})$ is called an *order ideal* if, for all $P \in A$ and $Q \subseteq P$, we also have $Q \in A$. An order ideal $A \in \mathscr{B}(\mathbb{A})$ is called a *principal ideal* if A has exactly one maximal element. In this case we may denote $A = \bar{P}$, where $P \in \mathscr{I}(\mathbb{A})$ is the unique maximum of A.

More generally, for (possibly non-convex) $P \in \mathscr{P}(\mathbb{A})$ denote

$$\bar{P} = \{Q \in \mathscr{I}(\mathbb{A}) \mid Q \subseteq P\},$$

that is, the collection of all convex $Q \subseteq P$ with vertices in $\mathbb{A}$. Note that, with the present terminology, the ideal $\bar{P}$ is *not* principal unless P is convex. The finiteness condition on $\mathscr{B}(\mathbb{A})$ guarantees that every ideal $A \in \mathscr{B}(\mathbb{A})$ is of the form $A = \bar{P}_1 \cup \cdots \cup \bar{P}_m$ for some finite collection of convex $P_1, ..., P_m \in \mathscr{I}(\mathbb{A})$.

A function $g: \mathscr{B}(\mathbb{A}) \to \mathbb{R}$ is called a *valuation* on $\mathscr{B}(\mathbb{A})$ if $g(\varnothing) = 0$, and

$$g(A \cup B) = g(A) + g(B) - g(A \cap B), \tag{6}$$

for all $A, B \in \mathscr{B}(\mathbb{A})$.

Given any function $f: \mathscr{I}(\mathbb{A}) \to \mathbb{R}$, one can define an *induced valuation* $\tilde{f}$ on the collection $\mathscr{B}(\mathbb{A})$ as follows. To begin, set $\tilde{f}(\varnothing) = 0$. For principal ideals $\bar{P}$, define

$$\tilde{f}(\bar{P}) = f(P).$$

Then for each $P \in \mathscr{I}(\mathbb{A})$ define $\tilde{f}$ on the singleton $\{P\}$ by

$$\tilde{f}(\{P\}) = \tilde{f}(\bar{P}) - \sum_{\substack{Q \in \bar{P} \\ Q \neq P}} \tilde{f}(\{Q\}).$$

It is well-known [20, 32] (and easy to prove by induction on the size of $P \cap \mathbb{A}$) that $\tilde{f}$ is a well-defined valuation on $\mathscr{B}(\mathbb{A})$. Moreover, for all $P \in \mathscr{I}(\mathbb{A})$ we have

$$f(P) = \tilde{f}(\bar{P}) = \sum_{Q \in \bar{P}} \tilde{f}(\{Q\}),$$

so that

$$\tilde{f}(\{P\}) = \sum_{Q \in \bar{P}} \mu(Q, P)\, \tilde{f}(\bar{Q}) = \sum_{Q \in \bar{P}} \mu(Q, P)\, f(Q), \tag{7}$$

by the Möbius inversion formula for partially ordered sets [31, 40].

3. POLYTOPE VALUATIONS

We now consider a different (but related) family of valuations. Denote by $\mathscr{I}^n$ the collection of *convex* polytopes in $\mathbb{R}^n$, and let $\mathscr{P}^n$ denote the collection of all (possibly non-convex) polytopes in $\mathbb{R}^n$; that is, the lattice of all *finite unions* of convex polytopes.

A function $\varphi: \mathscr{I}^n \to \mathbb{R}$ is called a *valuation on* $\mathscr{I}^n$ (or a *valuation on polytopes*) if $\varphi(\emptyset) = 0$, and

$$\varphi(P \cup Q) = \varphi(P) + \varphi(Q) - \varphi(P \cap Q), \tag{8}$$

for all $P, Q \in \mathscr{I}^n$ such that $P \cup Q \in \mathscr{I}^n$ as well. Groemer [14] has shown that every valuation on $\mathscr{I}^n$ has a unique extension to a valuation on all of $\mathscr{P}^n$. This extension is given by (8) for unions of pairs of convex polytopes, and by iterations of (8) for arbitrary finite unions of convex polytopes (see also [20]).

It is important to distinguish between the valuations on $\mathscr{B}(\mathbb{A})$ and valuations on $\mathscr{I}^n$ (or $\mathscr{P}^n$). While any *function* f on polytopes in $\mathscr{I}(\mathbb{A})$ induces a valuation $\tilde{f}$ on $\mathscr{B}(\mathbb{A})$ (as explained in the previous section), the original function f need not have been a valuation on the polytopes themselves (in the sense of (8)).

On the other hand, a *valuation on polytopes* φ also induces a valuation $\tilde{\varphi}$ on $\mathscr{B}(\mathbb{A})$. In this case, for all $P \in \mathscr{I}(\mathbb{A})$, we have $\tilde{\varphi}(\bar{P}) = \varphi(P)$.

This motivates a more simplified notation. For the remainder of this section, we use φ to denote both φ and $\tilde{\varphi}$. In other words, we use (7) to *extend* the valuation φ to all finite subsets of $\mathscr{I}(\mathbb{A})$ by setting $\varphi(\{P\}) = \tilde{\varphi}(\{P\})$ and

$$\varphi(P) = \sum_{Q \in \bar{P}} \varphi(\{Q\})$$

for all $P \in \mathscr{I}(\mathbb{A})$.

Polytope valuations differ from *measures* in that a valuation need not satisfy countable additivity; that is, additivity over countable unions (consider, for example, surface area, or the Euler characteristic). However, every valuation on $\mathscr{P}^n$ satisfies all *finite* inclusion-exclusion identities, and, like a measure, defines an integral (linear functional) on the space of simple functions on $\mathscr{P}^n$.

More specifically, for $P \in \mathscr{P}^n$ define the *indicator function* $1_P: \mathbb{R}^n \to \mathbb{R}$ by

$$1_P(x) = \begin{cases} 1 & \text{if} \quad x \in P \\ 0 & \text{if} \quad x \notin P \end{cases}$$

A *simple function* $f: \mathbb{R}^n \to \mathbb{R}$ on $\mathscr{P}^n$ is a finite linear combination of indicator functions of polytopes in $\mathscr{P}^n$. Recall that, for all $P, Q \in \mathscr{P}^n$, we have the *inclusion–exclusion identity*

$$1_P + 1_Q = 1_{P \cup Q} + 1_{P \cap Q}.$$

It follows that any linear functional on simple functions gives rise to a valuation on polytopes (by evaluating the function at indicator functions of polytopes).

The converse is also true. Given a valuation φ and a simple function $f = a_1 1_{P_1} + \cdots + a_m 1_{P_m}$ on $\mathscr{P}^n$, define the *integral of f with respect to φ* by

$$\int f \, d\varphi = \sum_{i=1}^{m} a_i \varphi(P_i). \tag{9}$$

The integral (9) is just a linear functional on the vector space generated by indicator functions of polytopes. Although a given simple function f may have many different expressions as linear combinations of indicator functions, Groemer [14] has shown that the integral expression (9) is well-defined (see also [20, p. 9]).

For a convex polytope P in $\mathbb{R}^n$ denote by $\operatorname{ri} P$ the *relative interior* of P (relative to its affine hull). Note that if $P \in \mathscr{P}^n$ is a topological manifold with boundary, then the boundary $\partial P \in \mathscr{P}^n$ as well, being the finite union of the facets of P. For any function $f\colon \mathscr{P}^n \to \mathbb{R}$ the value $f(\operatorname{ri} P)$ refers to the difference $f(P) - f(\partial P)$.

Suppose that $P \in \mathscr{P}^n$. For $x \in P$ define the *local Euler characteristic* $\chi(P, x)$ of P at x by

$$\chi(P, x) = \lim_{\varepsilon \to 0} \chi(P \cap B_\varepsilon(x)),$$

where $B_\varepsilon(x)$ denotes the *open* Euclidean ball of radius ε and centered at x. Note that if P is a convex polytope, then

$$\chi(P, x) = \begin{cases} (-1)^{\dim(P)} & \text{if } \; x \in \operatorname{ri} P \\ 0 & \text{otherwise} \end{cases} \tag{10}$$

since the Euler Characteristic of a relatively open n-ball is $\chi(B_\varepsilon) = (-1)^n$.

If we fix $x \in \mathbb{R}^n$, the functional $P \mapsto \chi(P, x)$ becomes a valuation in the parameter P, and we also have (from the definition of $\chi(P, x)$)

$$\chi(\operatorname{ri} P, x) = \begin{cases} (-1)^{\dim(P)} & \text{if } \; x \in P \\ 0 & \text{otherwise} \end{cases} \tag{11}$$

for all compact convex polytopes P.

For a valuation $\varphi\colon \mathscr{P}^n \to \mathbb{R}$ define the dual functional $\varphi^*\colon \mathscr{P}^n \to \mathbb{R}$ by

$$\varphi^*(P) = \int \chi(P, x) \, d\varphi. \tag{12}$$

Note that if $P \in \mathscr{P}^n$ is *convex*, then

$$\varphi^*(P) = (-1)^{\dim(P)} \varphi(\operatorname{ri} P),$$

by (10).

This value of this construction is demonstrated by the following theorem, due to Sallee [35].

THEOREM 3.1. *For all valuations $\varphi\colon \mathscr{P}^n \to \mathbb{R}$,*

1. *The functional φ^* is also a valuation.*
2. $\varphi^{**} = \varphi$.

Note. The mapping $\varphi \mapsto \varphi^*$ on valuations is also treated in the works of Sallee [35] and Macdonald [23] and corresponds to the mapping $[P] \mapsto [P]^*$ on the polytope algebra of McMullen [26]. See also [27, 28] for a discussion of this and related operators on the space of valuations.

Proof of Theorem 3.1. Since $\chi(P, x)$ is a valuation in the parameter P, and the integral with respect to φ in the expression (12) is additive, it follows that φ^* is a valuation.

When P is a compact convex potytope,

$$\varphi^{**}(P) = (-1)^{\dim(P)} \varphi^*(\operatorname{ri} P) = (-1)^{\dim(P)} \int \chi(\operatorname{ri} P, x)\, d\varphi.$$

From (11) we then have

$$\varphi^{**}(P) = (-1)^{\dim(P)} \int (-1)^{\dim(P)} 1_P(x)\, d\varphi = \varphi(P),$$

when P is a convex polytope. Since φ and φ^{**} are both valuations, it now follows from inclusion-exclusion arguments that $\varphi = \varphi^{**}$. ∎

A valuation φ on $\mathscr{P}^n$ is said to be *simple* if $\varphi(P) = 0$ whenever $\dim(P) < n$. Examples of simple valuations include Euclidean volume and all measures on $\mathbb{R}^n$ that are absolutely continuous with respect to volume. Evidently when φ is a *simple* valuation on $\mathscr{P}^n$ we have $\varphi^*(P) = (-1)^n \varphi(P)$ for all P.

4. A FREE POLYTOPE INVERSION IDENTITY

The following theorem characterizes polytope valuations in terms of the Euler-type relations.

THEOREM 4.1 (Inversion Theorem). *Let $\varphi: \mathscr{I}^n \to \mathbb{R}$ be a function on convex polytopes. The function φ extends to a valuation on $\mathscr{P}^n$ if and only if, for all locally finite sets $\mathbb{A} \subset \mathbb{R}^n$ and all $P \in \mathscr{I}(\mathbb{A})$, we have*

$$\varphi(\{P\}) = (-1)^{\dim(P)+1} \mu(\varnothing, P)\, \varphi(\text{ri } P)$$
$$= \begin{cases} (-1)^{\dim(P)+\alpha_1(P)+1} \varphi(\text{ri } P) & \text{if } P \text{ is free with respect to } \mathbb{A} \\ 0 & \text{otherwise} \end{cases}$$

where $\{P\}$ and μ are defined with respect to $\mathscr{I}(\mathbb{A})$ in each case.

The proof Theorem 4.1 is deferred to Section 5.

Using the language of the previous section, Theorem 4.1 can be rewritten:

$$\varphi(\{P\}) = -\mu(\varnothing, P)\, \varphi^*(P). \tag{13}$$

We can now state the main theorem of this article.

THEOREM 4.2 (Free Polytope Inversion Identity). *Let φ be a valuation on $\mathscr{P}^n$. For all $P \in \mathscr{I}(\mathbb{A})$, we have*

$$\varphi(P) = -\sum_{Q \subseteq P} \mu(\varnothing, Q)\, \varphi^*(Q). \tag{14}$$

Proof. From the definition of $\varphi(\{Q\})$ we have

$$\varphi(P) = \sum_{Q \subseteq P} \varphi(\{Q\}) = -\sum_{Q \subseteq P} \mu(\varnothing, Q)\, \varphi^*(Q),$$

where the final identity follows from (13). ∎

Theorem 4.2 is equivalent to the following reciprocal pair of inversion formulas for valuations on polytopes.

COROLLARY 4.3. *Let φ be a valuation on $\mathscr{P}^n$. For all $P \in \mathscr{I}(\mathbb{A})$, we have*

$$\varphi(P) = \sum_{Q \subseteq P} (-1)^{\dim(Q)+1} \mu(\varnothing, Q)\, \varphi(\text{ri } Q) \tag{15}$$

$$(-1)^{\dim(P)+1} \varphi(\text{ri } P) = \sum_{Q \subseteq P} \mu(\varnothing, Q)\, \varphi(Q) \tag{16}$$

where the sums in (15) and (16) are taken over all $Q \subseteq P$ such that $Q \in \mathscr{I}(\mathbb{A})$.

Proof. To prove (15), note once again that

$$\varphi(P) = \sum_{Q \subseteq P} \varphi(\{Q\}) = \sum_{Q \subseteq P} (-1)^{\dim(Q)+1} \mu(\varnothing, Q)\, \varphi(\text{ri } Q),$$

by Theorem 4.1. The identity (16) is then derived by applying (15) to the dual valuation φ^*. ∎

Recall that when φ is a simple valuation, $\varphi(P)=0$ whenever $\dim(P)<n$. In this case, we have $\varphi(P)=\varphi(\operatorname{ri} P)$ for all P. The two identities of Corollary 4.3 then coincide, taking the form:

$$(-1)^{\dim(P)+1}\,\varphi(P)=\sum_{Q\subseteq P}\mu(\varnothing, Q)\,\varphi(Q).$$

Important examples of simple valuations on polytopes include volume (Lebesgue measure), the Dehn invariant (used to solve Hilbert's 3rd Problem, see [34]), and restrictions of measures on $\mathbb{R}^n$ that are absolutely continuous with respect to Lebesgue measure. Some important families of simple valuations on compact convex sets in $\mathbb{R}^n$ are characterized in [16, 24, 25, 37] (see also [15, 20, 27]).

In analogy to the star of a point x in a polytope $P\in\mathscr{I}(\mathbb{A})$ we define the star of a polytope $M\subseteq P$ (relative to $\mathbb{A}$) as follows:

$$\mathrm{St}_P M=\{Q\in\mathscr{I}(\mathbb{A})\mid Q \text{ free and } M\subseteq Q\subseteq P\}.$$

By iterating the inversion formulas of Theorem 4.2 we obtain the following.

Corollary 4.4. *Let φ be a valuation on $\mathscr{P}^n$. For all $P\in\mathscr{I}(\mathbb{A})$, we have*

$$\varphi(P)=-\sum_{Q\subseteq P}\mu(\varnothing, Q)\,\varphi(Q)\,\chi(\mathrm{St}_P Q),$$

where we denote

$$\chi(\mathrm{St}_P Q)=\sum_{i=1}^{\infty}(-1)^i\,\alpha_i(\mathrm{St}_P Q)$$

Proof. On iterating Theorem 4.2 we obtain

$$\begin{aligned}\varphi(P)&=-\sum_{Q\subseteq P}\mu(\varnothing, Q)\,\varphi^*(Q)\\&=\sum_{Q\subseteq P}\mu(\varnothing, Q)\sum_{M\subseteq Q}\mu(\varnothing, M)\,\varphi(M)\\&=\sum_{M\subseteq P}\mu(\varnothing, M)\,\varphi(M)\sum_{M\subseteq Q\subseteq P}\mu(\varnothing, Q)\\&=-\sum_{M\subseteq P}\mu(\varnothing, M)\,\varphi(M)\,\chi(\mathrm{St}_P M),\end{aligned}$$

since

$$\chi(\mathrm{St}_P M) = \sum_{\substack{M \subseteq Q \subseteq P \\ Q \text{ free}}} (-1)^{\alpha_1(Q)+1} = -\sum_{M \subseteq Q \subseteq P} \mu(\emptyset, Q). \quad \blacksquare$$

Theorem 4.2 can be generalized to some examples of *non-convex* $P \in \mathscr{P}(\mathbb{A})$. Suppose that a polytope $P \in \mathscr{P}(\mathbb{A})$ can be decomposed into a union

$$P = Q_1 \cup \cdots \cup Q_m,$$

where the intersection $Q_{i_1} \cap \cdots \cap Q_{i_k} \in \mathscr{I}(\mathbb{A})$ for every subfamily $\{Q_{i_j}\} \subseteq \{Q_i\}$, and such that for every *free* polytope $M \subseteq P$, we have $M \subseteq Q_i$ for some i. In this case we call the collection $\{Q_i\}$ a *visible decomposition* of P. For example, the facets of a convex polytope $P \in \mathscr{I}(\mathbb{A})$ give a visible decomposition of the boundary ∂P.

THEOREM 4.5. *Let φ be a valuation on $\mathscr{P}^n$. Suppose $P \in \mathscr{P}(\mathbb{A})$ has a visible decomposition. Then*

$$\varphi(P) = \sum_{M \subseteq P} \varphi(\{M\}) = -\sum_{M \subseteq P} \mu(\emptyset, M)\, \varphi^*(M).$$

where the sum is taken over all $M \subseteq P$ such that $M \in \mathscr{I}(\mathbb{A})$.

Theorem 4.5 implies that the identities (15) and (16) hold for all $P \in \mathscr{P}(\mathbb{A})$ having a visible decomposition. Note that P need *not* be convex in Theorem 4.5, although the polytopes M in the sum are always convex.

Proof. Suppose that $P = Q_1 \cup \cdots \cup Q_m$ is a visible decomposition of P. For each $I \subseteq \{1, 2, ..., m\}$, denote

$$Q_I = \bigcap_{i \in I} Q_i.$$

Recall that we denote $\bar{Q}_I = \{M \in \mathscr{I}(\mathbb{A}) \mid M \subseteq Q_I\}$. From Theorem 4.1 we have $\varphi(\{M\}) = 0$ when M is not free. Since the collection $\{Q_i\}$ gives a visible decomposition of P, we have

$$\sum_{M \subseteq P} \varphi(\{M\}) = \sum_{M \in \bar{Q}_1 \cup \cdots \cup \bar{Q}_m} \varphi(\{M\}) = \sum_I (-1)^{|I|+1} \sum_{M \in \bar{Q}_I} \varphi(\{M\}),$$

by inclusion-exclusion. Since each $Q_I \in \mathscr{I}(\mathbb{A})$, we have $\varphi(Q_I) = \sum_{M \in \bar{Q}_I} \varphi(\{M\})$ for each I. It follows that

$$\sum_{M \subseteq P} \varphi(\{M\}) = \sum_{I} (-1)^{|I|+1} \varphi(Q_I). \tag{17}$$

Since φ is a valuation on polytopes, we apply inclusion-exclusion to the right hand side of (17) to obtain

$$\sum_{M \subseteq P} \varphi(\{M\}) = \varphi\left(\bigcup_{i=0}^{m} Q_i\right) = \varphi(P).$$

Hence,

$$\varphi(P) = \sum_{M \subseteq P} \varphi(\{M\}) = -\sum_{M \subseteq P} \mu(\emptyset, M)\, \varphi^*(M),$$

by Theorem 4.1. ∎

A (possibly non-convex) polytope $P \in \mathscr{P}(\mathbb{A})$ is said to be *non-singular* if P is also a topological manifold with (possibly empty) boundary ∂P. In this case we denote by $\operatorname{ri} P$ the relative interior of P, that is, $\operatorname{ri} P = P - \partial P$. Evidently every convex polytope is non-singular, and the boundary ∂P of any non-singular P is itself non-singular (with empty boundary).

Suppose $P \in \mathscr{P}(\mathbb{A})$ is non-singular and has a visible decomposition. Theorem 4.5 then implies that the identities (15) and (16) hold for the polytope P. From this point of view Theorems 4.2 and 4.5 describe free polytope analogues of Macdonald's relation for functionals on polytopes (which in turn generalize the Dehn–Sommerville equations) [23].

For example, consider the case where $P \in \mathscr{I}(\mathbb{A})$. The boundary ∂P is a topological sphere with empty boundary, $\partial(\partial P) = \emptyset$, so that $\operatorname{ri} \partial P = \partial P$ and $\dim(\partial P) + 1 = \dim(P)$. Moreover, the facets Q_i of P give a visible decomposition of ∂P. Theorem 4.5 (in the formulation of (16)) then implies that

$$(-1)^{\dim(P)}\, \varphi(\partial P) = \sum_{Q \subseteq \partial P} \mu(\emptyset, Q)\, \varphi(Q) \tag{18}$$

When P is a simplicial polytope, we can take $\mathbb{A}$ to be the vertices (extreme points) of P, so that the face enumerators $f_i = \alpha_{i+1}$ become valuations (on the face lattice of ∂P). On applying (18) to the case $\varphi = f_i$ we obtain the classical Dehn–Sommerville equations for the boundary of a convex polytope, namely,

$$(-1)^{\dim(P)}\, f_i(\partial P) = \sum_{k=i}^{\dim(P)-1} (-1)^{k+1} \binom{k+1}{i+1} f_k(\partial P),$$

for $i=0, ..., \dim(P)-1$. Similar considerations also yield a proof of Macdonald's more general Theorem 0.1.

5. THE PROOF OF THE INVERSION THEOREM

We now prove Theorem 4.1.

Given a regular cell complex K and a *subset* M of the geometric realization of K, denote by $\mathrm{Nb}_K(M)$ the set of all closed cells of K that intersect M. (Here "Nb" is for "neighborhood.") In other words,

$$\mathrm{Nb}_K(M)=\{\sigma\in K \mid \sigma\cap M\neq\varnothing\}.$$

When the context is clear, we may write "Nb" for "Nb_K."

The following lemmas treat important special cases of Theorem 4.1, leading in turn to its proof in the general case.

Lemma 5.1. *Suppose that Δ_n is an n-dimensional simplex in $\mathbb{R}^n$, and let ξ_k denote a k-dimensional flat in $\mathbb{R}^n$, then*

$$\chi(\mathrm{Nb}_{\Delta_n}(\xi_k\cap\Delta_n))=(-1)^{n-k}$$

if $\xi_k\cap\mathrm{int}(\Delta_n)\neq\varnothing$ and $\chi(\mathrm{Nb}_{\Delta_n}(\xi_k\cap\Delta_n))=0$ otherwise.

Proof. Suppose that $\xi_k\cap\mathrm{int}(\Delta_n)\neq\varnothing$. Let Z denote the subcomplex of Δ_n consisting of all faces of Δ_n that are disjoint from ξ_k. Evidently,

$$\chi(\mathrm{Nb}_{\Delta_n}(\xi_k\cap\Delta_n))=\chi(\Delta_n)-\chi(Z)=1-\chi(Z).$$

Refine the facial decomposition of Δ_n to form a new cell complex $\mathscr{C}$ in which the convex set $\Delta_n\cap\xi_k$ is a k-cell. The set Z of cells disjoint from ξ_k remains the *same* as in the original complex Δ_n. That is,

$$\chi(\mathrm{Nb}_{\mathscr{C}}(\xi_k\cap\Delta_n))=1-\chi(Z),$$

where $\mathrm{Nb}_{\mathscr{C}}$ denotes the neighborhood with respect to the cell complex $\mathscr{C}$. Hence, $\chi(\mathrm{Nb}_{\Delta_n}(\xi_k\cap\Delta_n))=\chi(\mathrm{Nb}_{\mathscr{C}}(\xi_k\cap\Delta_n))$. But $Z=\mathrm{Lk}_{\mathscr{C}}(\xi_k\cap\Delta_n)$, the *link* in Δ_n with respect to $\mathscr{C}$. Since $\xi_k\cap\Delta_n$ is an interior k-cell of the homology manifold (with boundary) $\mathscr{C}$, it is well-known (see, for example, [29]) that $\chi(Z)=1-(-1)^{n-k}$. Therefore,

$$\chi(\mathrm{Nb}_{\Delta_n}(\xi_k\cap\Delta_n))=\chi(\mathrm{Nb}_{\mathscr{C}}(\xi_k\cap\Delta_n))=1-\chi(Z)=(-1)^{n-k}.$$

A similar argument applies to the boundary case. ∎

The following lemma deals with the free case of Theorem 4.1. The argument used to prove this lemma will anticipate the one to be used in the more general case.

LEMMA 5.2. *Suppose φ is a valuation on $\mathscr{P}^n$, and suppose that $P \in \mathscr{I}(\mathbb{A})$. If P is free, then*

$$\varphi(\{P\}) = (-1)^{\dim(P)+1}\, \mu(\emptyset, P)\, \varphi(\mathrm{ri}\, P).$$

Proof. From the Möbius inversion formula we have

$$\varphi(\{P\}) = \sum_{Q \subseteq P} \mu(Q, P)\, \varphi(Q) = \sum_{Q \subseteq P} \mu(Q, P) \int_{\mathbb{R}^n} 1_Q(x)\, d\varphi = \int_{\mathbb{R}^n} f_P(x)\, d\varphi,$$

where

$$f_P(x) = \sum_{Q \subseteq P} \mu(Q, P)\, 1_Q(x) = \sum_{\{x\} \subseteq Q \subseteq P} \mu(Q, P).$$

Since P is free, each $Q \subseteq P$ in $\mathscr{I}(\mathbb{A})$ is also free. It then follows from Proposition 1.1 that $\mu(Q, P) = (-1)^{\alpha_1(P) - \alpha_1(Q)} = \mu(\emptyset, P)(-1)^{\alpha_1(Q)}$.

Since P is a free polytope having $\alpha_1(P)$ vertices, the visibility complex $\mathrm{Vis}(P)$ is a simplex of dimension $\alpha(P) - 1$. Let π: $\mathrm{Vis}(P) \to P$ denote the projection of $\mathrm{Vis}(P)$ onto P mapping faces of $\mathrm{Vis}(P)$ onto free polytopes inside P.

For $x \in P$, the pre-image $\pi^{-1}(x)$ in $\mathrm{Vis}(P)$ is of the form $\pi^{-1}(x) = \mathrm{Vis}(P) \cap \xi$, where ξ is a flat of dimension $\alpha_1(P)$ in $\mathbb{R}^{n+\alpha_1(P)}$. Moreover, ξ meets the interior of $\mathrm{Vis}(P)$ if and only if $x \in \mathrm{ri}\, P$.

Free polytopes Q on i vertices inside P such that $x \in Q$ are in one-to-one correspondence with $(i-1)$-dimensional faces of $\mathrm{Vis}(P)$ that meet the flat ξ. Suppose $x \in \mathrm{ri}\, P$. While ξ is a flat of dimension $\alpha_1(P)$ in $\mathbb{R}^{n+\alpha_1(P)}$, the flat ξ meets the affine hull of $\mathrm{Vis}(P)$ (and, therefore, $\mathrm{ri}(\mathrm{Vis}(P))$) in dimension $\alpha_1(P) - \dim(P) - 1$. We then have

$$\begin{aligned}
\chi(\mathrm{Nb}_{\mathrm{Vis(P)}}(\xi \cap P)) &= (-1)^{\dim(\mathrm{Vis}(P)) - \dim(\xi \cap \mathrm{Aff}(\mathrm{Vis}(P)))} \\
&= (-1)^{(\alpha_1(P)-1) - (\alpha_1(P) - \dim(P) - 1)} \\
&= (-1)^{\dim(P)}
\end{aligned}$$

by Lemma 5.1. A similar argument in the boundary case implies that

$$\chi(\mathrm{Nb}_{\mathrm{Vis(P)}}(\xi \cap P)) = 0$$

when $x \in \partial P$.

We then have

$$
\begin{aligned}
f_P(x) &= \mu(\emptyset, P) \sum_{\{x\} \subseteq Q \subseteq P} (-1)^{\alpha_1(Q)} \\
&= -\mu(\emptyset, P)\, \chi(\mathrm{Nb}_{\mathrm{Vis}(\mathbf{P})}(\xi \cap P)) \\
&= \mu(\emptyset, P)(-1)^{\dim(P)+1},
\end{aligned}
$$

if $x \in \mathrm{ri}\, P$, while $f_P(x) = 0$ otherwise. Hence,

$$\varphi(\{P\}) = (-1)^{\dim(P)+1}\, \mu(\emptyset, P)\, \varphi(\mathrm{ri}\, P). \quad \blacksquare$$

Suppose $P \in \mathscr{I}(\mathbb{A})$ is not free. Let $\mathrm{Ext}(P)$ denote the set of extreme points of P (all of which are points of $\mathbb{A}$). Since P is not free, the set $P \cap (\mathbb{A} - \mathrm{Ext}(P)) \neq \emptyset$. Let P_0 be the convex hull of $P \cap (\mathbb{A} - \mathrm{Ext}(P))$, that is, of the points of $P \cap \mathbb{A}$ that are not extreme in P.

LEMMA 5.3. *Suppose φ is a valuation on $\mathscr{P}^n$, suppose that $P \in \mathscr{I}(\mathbb{A})$. If P is not free, then $\varphi(\{P\}) = 0$.*

Proof. Since P is not free, we have $P_0 \neq \emptyset$. Recall from Proposition 1.1 that $\mu(Q, P) = 0$ unless $P_0 \subseteq Q$. From the Möbius inversion formula we have

$$\varphi(\{P\}) = \sum_{Q \subseteq P} \mu(Q, P)\, \varphi(Q) = \sum_{P_0 \subseteq Q \subseteq P} \mu(Q, P) \int_{\mathbb{R}^n} 1_Q(x)\, d\varphi = \int_{\mathbb{R}^n} g_P(x)\, d\varphi,$$

where

$$g_P(x) = \sum_{P_0 \subseteq Q \subseteq P} \mu(Q, P)\, 1_Q(x) = \sum_{\substack{P_0 \subseteq Q \subseteq P \\ x \in Q}} \mu(Q, P).$$

We now show that $g_P(x) = 0$ for all $x \in P$.

Recall that, for $P \in \mathscr{I}(\mathbb{A})$, we denote by $\Delta(P)$, or just Δ, the simplex of dimension $\alpha_1(P) - 1$ consisting of all subsets of $P \cap \mathbb{A}$. For all faces $S \subseteq \Delta$ denote by $\mathrm{St}\, S$ (resp. $\overline{\mathrm{St}}\, S$) the open (resp. closed) star of S in Δ. Similar, let $\mathrm{Lk}\, S$ denote the link of S in Δ.

Let ξ denote the pre-image of the point x under the projection map $\pi\colon \Delta(P) \to P$. Then we have

$$\begin{aligned}
g_P(x) &= \sum_{x \in Q \subseteq P} \mu(Q, P) \\
&= \sum_{\substack{P_0 \subseteq Q \subseteq P \\ x \in Q}} (-1)^{\alpha_1(P) - \alpha_1(Q)} \\
&= \sum_{\substack{Q \in \mathrm{St}\, P_0 \\ \xi \cap Q \neq \varnothing}} (-1)^{\alpha_1(P) - \alpha_1(Q)} \\
&= (-1)^{\alpha_1(P)} \left(\sum_{\substack{Q \in \varDelta \\ \xi \cap Q \neq \varnothing}} \alpha_1(Q) - \sum_{\substack{Q \in \varDelta - \mathrm{St}\, P_0 \\ \xi \cap Q \neq \varnothing}} \alpha_1(Q) \right) \\
&= (-1)^{\alpha_1(P)+1} \left(\chi(\mathrm{Nb}_\varDelta\, \xi) - \chi(\mathrm{Nb}_{\mathrm{Lk}\, P_0 * \partial P_0}\, \xi)\right).
\end{aligned}$$

Recall from Lemma 5.1 that

$$\chi(\mathrm{Nb}_\varDelta\, \xi) = \begin{cases} (-1)^{\dim(\varDelta) - \dim(\xi \cap \mathrm{Aff}(\varDelta))} = (-1)^{\dim(P)} & \text{if } \quad \xi \cap \mathrm{ri}(\varDelta) \neq \varnothing \\ 0 & \text{otherwise} \end{cases}$$

Meanwhile, denote by $u_1, ..., u_m$ the points of $P_0 \cap \mathbb{A}$, and let σ_i denote the facet of $\varDelta$ consisting of

$$\sigma_i = \tilde{P} \vee u_1 \vee \cdots u_{i-1} \vee u_{i+1} \vee \cdots \vee u_m,$$

where $\tilde{P}$ denotes the face of $\varDelta$ spanned by the extreme points of P, and where $\vee$ denotes convex hull inside $\varDelta$. We then have

$$\mathrm{Lk}\, P_0 * \partial P_0 = \bigcup_{i=1}^{m} \sigma_i.$$

Denote $[m] = \{1, 2, ..., m\}$. For $I \subseteq [m]$ let

$$\sigma_I = \bigcap_{i \in I} \sigma_i = \tilde{P} \vee \bigvee_{j \notin I} u_i.$$

For each I we have

$$(\mathrm{Nb}_{\mathrm{Lk} P_0 * \partial P_0}\, \xi) \cap \sigma_I = \mathrm{Nb}_{\sigma_I} \xi.$$

Since χ is a valuation on subcomplexes of $\varDelta$, the inclusion-exclusion identity (1), applied to the union

$$\mathrm{Nb}_{\mathrm{Lk} P_0 * \partial P_0}\, \xi = \bigcup_{i=1}^{m} \mathrm{Nb}_{\sigma_i}\, \xi,$$

implies that

$$\chi(\mathrm{Nb}_{\mathrm{Lk}P_0 * \partial P_0}\,\xi) = \sum_{i=1}^{m} \chi((\mathrm{Nb}_{\mathrm{Lk}P_0 * \partial P_0}\,\xi) \cap \sigma_i)$$

$$- \sum_{i<j} \chi((\mathrm{Nb}_{\mathrm{Lk}P_0 * \partial P_0}\,\xi) \cap \sigma_i \cap \sigma_j) + \cdots$$

$$+ (-1)^m \chi((\mathrm{Nb}_{\mathrm{Lk}P_0 * \partial P_0}\,\xi) \cap \sigma_1 \cap \cdots \cap \sigma_m)$$

$$= \sum_{\varnothing \neq I \subseteq [m]} (-1)^{|I|+1} \chi(\mathrm{Nb}_{\sigma_I}\,\xi). \tag{19}$$

If $x \in \partial P$ then $\xi \cap \sigma_I \subseteq \partial \sigma_I$ for all I, since $\xi \cap \mathrm{ri}(\tilde{P}) = \varnothing$. Therefore, all terms of (19) are zero, by Lemma 5.1.

If $x \in \mathrm{ri}\, P$, then $\xi \cap \sigma_I$ meets $\mathrm{ri}\, \sigma_I$ for all I, so that each term of (19) is $(-1)^{\dim P}$. In this case we have

$$\chi(\mathrm{Nb}_{\mathrm{Lk}P_0 * \partial P_0}\,\xi) = \sum_{\varnothing \neq I \subseteq [m]} (-1)^{|I|+1} (-1)^{\dim P}$$

$$= (-1)^{\dim P} \sum_{i=1}^{m} \binom{m}{i} (-1)^{i+1} = (-1)^{\dim P}.$$

Hence,

$$g_P(x) = (-1)^{\alpha_1(P)+1} (\chi(\mathrm{Nb}_{\Delta}\,\xi) - \chi(\mathrm{Nb}_{\mathrm{Lk}P_0 * \partial P_0}\,\xi))$$

$$= \left.\begin{cases} (-1)^{\alpha_1(P)+1}(0-0) = 0 & \text{if } \ x \in \partial P \\ (-1)^{\alpha_1(P)+1}((-1)^{\dim P} - (-1)^{\dim P}) = 0 & \text{if } \ x \in \mathrm{ri}\, P \end{cases}\right\} = 0.$$

This completes the proof of Lemma 5.3. ∎

We have now done most of the work needed to proof the Inversion Theorem 4.1.

Proof of Theorem 4.1. Suppose that $\varphi: \mathscr{P}^n \to \mathbb{R}$ is a valuation. If $P \in \mathscr{I}(\mathbb{A})$ is free, then

$$\varphi(\{P\}) = (-1)^{\dim(P)+1} \mu(\varnothing, P)\, \varphi(\mathrm{ri}\, P),$$

by Lemma 5.2. If $P \in \mathscr{I}(\mathbb{A})$ is not free then $\mu(\varnothing, P) = 0$. The theorem then follows from Lemma 5.3.

Next suppose that $\varphi\colon \mathscr{I}^n \to \mathbb{R}$ is a function, and suppose also that, for all locally finite sets $\mathbb{A} \subset \mathbb{R}^n$ we have

$$\varphi(\{P\}) = (-1)^{\dim(P)+1} \mu(\varnothing, P)\, \varphi(\mathrm{ri}\, P), \tag{20}$$

where $\{P\}$ and μ are defined with respect to $\mathscr{I}(\mathbb{A})$ for each $\mathbb{A}$. We show that φ extends to a unique valuation on $\mathscr{P}^n$.

To begin, recall once again the theorem of Groemer [14] (see also [20]) that a function f on $\mathscr{I}^n$ has a unique extension to a *valuation* on $\mathscr{P}^n$ if and only if

$$f(P_1 \cup P_2) + f(P_1 \cap P_2) = f(P_1) + f(P_2), \tag{21}$$

whenever P_1, P_2, and $P_1 \cup P_2$ are *convex* polytopes. We show that the identity (21) holds for the function φ.

Suppose $P_1, P_2 \in \mathscr{I}^n$ and suppose that $P_1 \cup P_2$ is also convex. Let $\mathbb{A}'$ denote the collection of all extreme points of P_1 and of P_2. If $P_1 \cap P_2 \neq \varnothing$ fill $P_1 \cap P_2$ with a finite collection points $\mathbb{A}''$ so that every edge, triangle, and simplex of any dimension having extreme points in *both* P_1 and P_2 will contain a relative interior point in $\mathbb{A}''$. If $P_1 \cap P_2 = \varnothing$ let $\mathbb{A}'' = \varnothing$. In either case, let $\mathbb{A} = \mathbb{A}' \cup \mathbb{A}''$.

It follows that $P_1, P_2, P_1 \cup P_2, P_1 \cap P_2 \in \mathscr{I}(\mathbb{A})$. Moreover, if $Q \subseteq P_1 \cup P_2$ is free with respect to $\mathbb{A}$, then either $Q \subseteq P_1$ or $Q \subseteq P_2$ or both. From (20) we then have

$$\begin{aligned}
\varphi(P_1 \cup P_2) &= \sum_{Q \subseteq P_1 \cup P_2} \varphi(\{Q\}) \\
&= \sum_{\substack{Q \subseteq P_1 \cup P_2 \\ Q \text{ free}}} \varphi(\{Q\}) \\
&= \sum_{\substack{Q \subseteq P_1 \\ Q \text{ free}}} \varphi(\{Q\}) + \sum_{\substack{Q \subseteq P_2 \\ Q \text{ free}}} \varphi(\{Q\}) - \sum_{\substack{Q \subseteq P_1 \cap P_2 \\ Q \text{ free}}} \varphi(\{Q\}) \\
&= \varphi(P_1) + \varphi(P_2) - \varphi(P_1 \cap P_2),
\end{aligned}$$

so that φ is a valuation on polytopes. ∎

6. SOME EXAMPLES AND APPLICATIONS

We now apply the identities of Section 4 to some important examples of valuations on polytopes.

6.1. *Measures and Volume*

A common family of examples arises from the case in which $\varphi = \lambda$, where λ is a (signed) *measure* on $\mathbb{R}^n$. We then have $\lambda(P) - \lambda(\partial P) = \lambda(\mathrm{ri}\, P)$, for all convex polytopes $P \in \mathscr{I}(\mathbb{A})$. Theorem 4.2 then implies that

$$\lambda(\mathrm{ri}\, P) = (-1)^{\dim(P)+1} \sum_{\substack{Q \subseteq P \\ Q \text{ free}}} (-1)^{\alpha_1(Q)}\, \lambda(Q).$$

If λ is absolutely continuous with respect to Lebesgue measure, then $\lambda(\partial P) = 0$ for all P, and $\lambda(\mathrm{ri}\, P) = \lambda(P)$, so that

$$\lambda(P) = (-1)^{n+1} \sum_{\substack{Q \subseteq P \\ Q \text{ free}}} (-1)^{\alpha_1(Q)}\, \lambda(Q),$$

as in the more general case of simple valuations.

An important special case is when $\lambda = V_n$, the Euclidean volume on $\mathbb{R}^n$. In this case Theorem 4.5 takes the following appealing form.

PROPOSITION 6.1 (Volume Formula). *Let $\mathbb{A}$ be a locally finite subset of $\mathbb{R}^n$. For all $P \in \mathscr{P}(\mathbb{A})$ having a visible decomposition,*

$$V_n(P) = (-1)^{n+1} \sum_{\substack{Q \subseteq P \\ Q \text{ free}}} (-1)^{\alpha_1(Q)}\, V_n(Q)$$

This formula simplifies even further in the special case of *area* for polygons having vertices in $\mathbb{A} = \mathbb{Z}^2$. See Section 7 and also [19].

6.2. *Surface Area*

Recall that every convex polytope in $\mathbb{R}^n$ has a well-defined *surface area*, denoted $S(P)$. If we set $S(P) = 2V_{n-1}(P)$ for polytopes of dimension $n-1$ in $\mathbb{R}^n$, then S extends to a valuation on all of $\mathscr{P}^n$. For convex polytopes P of dimension n, the boundary ∂P has dimension $n-1$, so that $S(\partial P) = 2V_{n-1}(\partial P) = 2S(P)$. It follows that

$$S(P) - S(\partial P) = S(P) - 2S(P) = -S(P).$$

Theorem 4.2 then implies that

$$S(P) = (-1)^n \sum_{\substack{Q \subseteq P \\ Q \text{ free}}} (-1)^{\alpha_1(Q)}\, S(Q), \tag{22}$$

for all convex polytopes in $\mathscr{I}(\mathbb{A})$ having non-empty interior in $\mathbb{R}^n$.

Remark. If $\mathbb{A} = \mathbb{Z}^n$, the set of integral points of $\mathbb{R}^n$, then the previous formula can be restated with ordinary surface area replaced by the *relative surface area* for integral polytopes.

6.3. *The Euler Characteristic*

It is well-known that the Euler characteristic $\chi(P)$ of a (possibly non-convex) polytope gives the unique valuation on $\mathscr{P}^n$ taking the value 1 on all non-empty compact *convex* polytopes. See, for example, [20] for a detailed construction. Recall also that if Q is a non-empty compact convex polytope, then $\chi(\mathrm{ri}\ Q) = (-1)^{\dim Q}$, so that $\chi^* = \chi$. On applying Theorem 4.5 to the Euler characteristic we now obtain the following result.

PROPOSITION 6.2 (Free Polytope Euler Formula). *For all $P \in \mathscr{P}(\mathbb{A})$ having a visible decomposition,*

$$\chi(P) = -\sum_{\varnothing \neq Q \subseteq P} \mu(\varnothing, Q) = \sum_{i=1}^{\infty} (-1)^{i+1} \alpha_i(P). \tag{23}$$

Note that the sums in (23) are always finite, since every $P \in \mathscr{P}(\mathbb{A})$ contains a finite number of polytopes $Q \in \mathscr{I}(\mathbb{A})$. The identity (23) is comparable to the classical Euler formula, which gives the Euler characteristic of P as the alternating sum of the face numbers of a specific cell decomposition of a polytope [20, 29, 43].

While it is tempting to conjecture that (23) holds for *all* polytopes $P \in \mathscr{P}(\mathbb{A})$, this turns out to be false. Let Δ_3 denote a 3-dimensional regular tetrahedron. Choose a pair of disjoint edges of Δ_3, and attach a triangular fin to each edge from this pair, to form a (non-convex) polytope M. Adjust the fins so that the line segment connecting the outer tip of each fin lies inside M. Now let $\mathbb{A}$ denote the vertices of Δ_3 along with the outer tips of each of the two added fins, a total of 6 points. Clearly $M \in \mathscr{P}(\mathbb{A})$ and M is contractible, so that $\chi(M) = 1$. However, we also have $\alpha_1(M) = 6$, $\alpha_2(M) = 11$, $\alpha_3(M) = 6$, $\alpha_4(M) = 1$, while $\alpha_i(M) = 0$ for $i \geqslant 5$. Hence,

$$\sum_{i=1}^{\infty} (-1)^{i+1} \alpha_i(M) = 0 \neq \chi(M).$$

This counterexample to the universality of (23) is due to Edelman and Reiner [10].

6.4. *Interior Points*

The functional α_1, which counts the points of $\mathbb{A}$ contained inside a polytope P, is a valuation on $\mathscr{P}^n$. This follows from the fact that

$$\begin{aligned}\alpha_1(P \cup Q) + \alpha_1(P \cap Q) &= |(P \cup Q) \cap \mathbb{A}| + |(P \cap Q) \cap \mathbb{A}| \\ &= |P \cap \mathbb{A}| + |Q \cap \mathbb{A}| \\ &= \alpha_1(P) + \alpha_1(Q). \end{aligned} \tag{24}$$

For non-singular polytopes P, define the related functional

$$I(P) = |\mathrm{ri}(P) \cap \mathbb{A}| = \alpha_1(P) - \alpha_1(\partial P),$$

which counts the number of points of $\mathbb{A}$ contained in the *relative interior* of P. Evidently $I(P) = (-1)^{\dim(P)} \alpha_1^*(P)$. Theorem 4.5 therefore yields the following formula for $I(P)$.

PROPOSITION 6.3. *For all non-singular $P \in \mathscr{P}(\mathbb{A})$ having a visible decomposition,*

$$I(P) = \sum_{i=1}^{\infty} (-1)^{\dim(P)+i+1}\, i\alpha_i(P). \tag{25}$$

Evidently the sum on the right-hand side of (25) is always finite, since any given P contains a finite number of free polytopes $Q \in \mathscr{I}(\mathbb{A})$.

Ahrens, Gordon, and McMahon [1] independently discovered a 2-dimensional version of Proposition 6.3 for convex polygons, and they posed the n-dimensional version as a conjecture. The generalization of (25) to $\mathbb{R}^n$ for convex polytopes was also proved independently by Edelman and Reiner [9].

Proof of Proposition 6.3. Since α_1 is a valuation on $\mathscr{P}^n$, we apply Theorem 4.5 to the valuation α_1^* to obtain

$$\begin{aligned}\alpha_1^*(P) = -\sum_{Q \subseteq P} \mu(\varnothing, Q)\, \alpha_1(Q) &= -\sum_{\substack{Q \subseteq P \\ Q \text{ free}}} (-1)^{\alpha_1(Q)}\, \alpha_1(Q) \\ &= \sum_{i=1}^{\infty} (-1)^{i+1}\, i\alpha_i(P).\end{aligned}$$

Since $I(P) = (-1)^{\dim(P)} \alpha_1^*(P)$, the identity (25) immediately follows. ∎

Proposition 6.3 is an example of a *Dehn–Sommerville equation.* While any particular cell decomposition of a convex polytope satisfies a family of very general Dehn–Sommerville equations (see, for example, [2, 5, 6, 13, 21, 40, 42, 43]), the visibility complex of a convex polytope is typically too complicated to admit Dehn–Sommerville equations for the higher dimensional enumerators α_i, where $i > 1$. In particular, $\mathrm{Vis}(P)$ usually is *not* an Eulerian manifold, as defined by Klee [21].

Moreover, the free polytope enumerators α_i typically are *not* valuations for $i > 1$, so that Theorem 4.2 does not typically apply to them. For example, the identity (24) fails to hold for the free edge enumerator α_2, even for some of the simplest examples.

7. INTEGRAL POLYTOPES AND A FORMULA FOR AREA

Let $\mathbb{Z}^n$ denote the set of all points in $\mathbb{R}^n$ having integer coordinates. Denote by $\mathcal{I}^n = \mathcal{I}(\mathbb{Z}^n)$ the set of all convex polytopes having all of their vertices in $\mathbb{Z}^n$. Elements of $\mathcal{I}^n$ will also be referred to as convex *integral polytopes*. The results of the preceding sections take especially simple and elegant forms for the 2-dimensional case, in which $\mathbb{A} = \mathbb{Z}^2$, as a consequence of the following proposition.

PROPOSITION 7.1. *Let $P \in \mathcal{I}^2$. If P is free then P is either a point, a line segment, a free triangle of area* 1/2, *or a free parallelogram of area* 1.

In particular, free convex polygons have area 0, 1/2, or 1, and any free convex polygon has an affine unimodular image inside the unit square.

Proof. From the tiling properties of parallelograms (and some elementary linear algebra) it follows that a parallelogram in $\mathcal{I}^2$ is free if and only if it is the image of a translate of the unit square under a unimodular transformation. In other words, a parallelogram is free if and only if it has unit area. Since any free triangle forms a free parallelogram when pasted to its reflection through the center of an edge, an integral triangle is free if and only if has area 1/2.

Let P be an arbitrary free convex polygon with at least four vertices. Any three vertices of P span a free triangle inside P. Let x_1, x_2, x_3 denote 3 adjacent vertices of P such that $\overline{x_1x_2}$ and $\overline{x_2x_3}$ are boundary edges of P. There exist a translation and a unimodular transformation mapping these vertices x_1, x_2, x_3 to the points $y_1 = (1, 0)$, $y_2 = (0, 0)$, and $y_3 = (0, 1)$ respectively. Let Q denote the image of P under this transformation. Note that Q is also free, and that the edges $\overline{y_1y_2}$ and $\overline{y_2y_3}$ are boundary edges of Q. Since Q is convex and free, all remaining vertices of Q must have positive (integer) coordinates.

Let (a, b) be a fourth vertex of Q. If $a > 1$, then the triangle with vertices $(0, 0)$, $(0, 1)$, and (a, b) has area greater than 1/2, and is consequently *not* free. This contradicts the assumption that Q is free. Therefore $a \leqslant 1$. Similarly, $b \leqslant 1$, and Q must be the unit square. It follows that P must be a free parallelogram. ∎

It follows from Proposition 7.1 that

$$\alpha_3(P) = \text{number of free integral triangles inside } P,$$

while

$$\alpha_4(P) = \text{number of free integral parallelograms inside } P,$$

for all $P \in \mathscr{P}(\mathbb{Z}^2)$. Evidently when $k \geqslant 5$ we have $\alpha_k(P) = 0$ for all $P \in \mathscr{P}(\mathbb{Z}^2)$.

Let $A(P)$ denote the *area* of a polygon P. Theorem 4.2 and its corollaries now yield the following formula for the area of an integral polygon.

THEOREM 7.2 (Area Formula). *For all polygons $P \in \mathscr{P}(\mathbb{Z}^2)$ having a visible decomposition,*

$$A(P) = (1/2)\,\alpha_3(P) - \alpha_4(P).$$

Proof. Viewing the area A as two-dimensional volume V_2, apply Proposition 6.1 to obtain

$$A(P) = \sum_{\substack{Q \subseteq P \\ Q \text{ free}}} (-1)^{\alpha_1(Q)+1} A(Q) = (1/2)\,\alpha_3(P) - \alpha_4(P),$$

where the second equality follows from Proposition 7.1. ∎

Consider, for example, the polygon (having area $\frac{5}{2}$) in Fig. 1 of Section 1. A more elementary (and purely combinatorial) proof of this area formula for the case of convex integral polygons can be found in [19].

For 2-dimensional non-singular polygons $P \in \mathscr{P}(\mathbb{Z}^2)$, define $B(P) = \alpha_1(P) - I(P)$, the number of lattice points on the boundary ∂P of P. On combining Proposition 6.3 with Proposition 7.1 we obtain the following formulas for the number of interior and boundary integral points of a polygon (see also [19]).

PROPOSITION 7.3. *For all non-singular polygons $P \in \mathscr{P}(\mathbb{Z}^2)$ having a visible decomposition and non-empty interior,*

$$I(P) = \alpha_1(P) - 2\alpha_2(P) + 3\alpha_3(P) - 4\alpha_4(P). \tag{26}$$

and

$$B(P) = 2\alpha_2(P) - 3\alpha_3(P) + 4\alpha_4(P). \tag{27}$$

Proof. Equation (26) follows immediately from Propositions 7.1 and 6.3. Equation (27) then follows from (26) and the fact that $I(P) + B(P) = \alpha_1(P)$. ∎

Evidently the values $\alpha_3(P)$ and $\alpha_4(P)$ take some considerable effort to compute, even assuming knowledge of which integral points lie inside and on the boundary of P. Some of this effort is saved if we invert the relations above (using elementary linear algebra) to obtain the following.

COROLLARY 7.4. *For all non-singular polygons $P \in \mathscr{P}(\mathbb{Z}^2)$ having a visible decomposition and non-empty interior,*

$$\alpha_3(P) = -3\alpha_1(P) + 2\alpha_2(P) - I(P) + 4\chi(P)$$

$$\alpha_4(P) = -2\alpha_1(P) + \alpha_2(P) - I(P) + 3\chi(P)$$

The following well-known result is an immediate consequence of Theorem 7.2, Proposition 7.3 and the Euler characteristic formula (5) (see also [12, 13, 30]).

COROLLARY 7.5 (Pick's Theorem). *For all non-singular polygons $P \in \mathscr{P}(\mathbb{Z}^2)$ having a visible decomposition and non-empty interior,*

$$A(P) = I(P) + (1/2)\, B(P) - \chi(P).$$

Although Theorem 4.5 applies (as stated in this article) only to functionals on polytopes having visible decomposition, the results of this section (for polygons in $\mathscr{I}(\mathbb{Z}^2)$) can be generalized to all (non-convex) polygons $P \in \mathscr{P}(\mathbb{Z}^2)$ by means of other combinatorial arguments (as found in [19], for example).

The situation is far more complicated for integral polytopes of dimension $n \geqslant 3$. On the one hand, by considering vertex coordinates modulo 2 one can show that any free convex integral polytope in $\mathbb{R}^n$ has at most 2^n vertices. Unfortunately, Proposition 7.1 has no finite analogue in higher dimensions; even in dimension 3 there exist *free* integral simplices (free polytopes with exactly 4 integral points) of arbitrarily large volume. As a result, the analogue of Theorem 7.2 for 3-dimensional volume will potentially involve a finite, but arbitrarily large, family of free polytope enumerators; as might Proposition 6.1 for general vertex sets $\mathbb{A}$.

8. DUAL RELATIONS AND INCLUSION–EXCLUSION IDENTITIES

On combining Theorem 4.1 with Möbius inversion we obtain the following relation dual to the inversion identity of Theorem 4.2.

THEOREM 8.1. *Let φ be a valuation on $\mathscr{P}^n$. For all $P \in \mathscr{I}(\mathbb{A})$, we have*

$$\mu(\varnothing, P)\, \varphi^*(P) = -\sum_{Q \subseteq P} \mu(Q, P)\, \varphi(Q) \tag{28}$$

Note that we require the polytope P to be convex in Theorem 8.1. Since $\varphi^{**} = \varphi$, we can also exchange the roles of φ and φ^* in (28).

Theorem 8.1 is dual (in yet another sense) to the free polytope inversion identity of Theorem 4.2, in that Möbius functions from minimal elements $\mu(\varnothing, \cdot)$ are replaced with Möbius functions to maximal elements $\mu(\cdot, P)$.

Proof. On combining Möbius inversion formula (7) and Theorem 4.1, we obtain

$$\mu(\varnothing, P)\, \varphi^*(P) = -\varphi(\{P\}) = -\sum_{Q \subseteq P} \mu(Q, P)\, \varphi(Q). \quad \blacksquare$$

Theorem 8.1 also has some striking consequences for the case of polytopes in $\mathscr{I}(\mathbb{A})$ that are not free.

COROLLARY 8.2. *If $P \in \mathscr{I}(\mathbb{A})$ is not free, then*

$$\sum_{Q \subseteq P} \mu(Q, P)\, \varphi(Q) = 0.$$

Proof. This is an immediate consequence of Theorem 8.1 and the fact that $\mu(\varnothing, P) = 0$ when P is not free (see Proposition 1.1). $\blacksquare$

In particular, suppose $P \in \mathscr{I}(\mathbb{A})$ is not free, and let $x_1, ..., x_m$ denote the extreme points of P (where $x_i \in \mathbb{A}$). Since P is not free, $P \cap (\mathbb{A} - \{x_1, ..., x_m\}) \neq \varnothing$. Let P_0 denote the convex hull of $P \cap (\mathbb{A} - \{x_1, ..., x_m\})$; that is, the convex hull of the points of $P \cap \mathbb{A}$ that are not extreme in P. Recall from Proposition 1.1 that, for $Q \subseteq P$, we have $\mu(Q, P) = 0$ unless $P_0 \subseteq Q$. From Corollary 8.2 and $\mu(P, P) = 1$ it follows that

$$\varphi(P) = -\sum_{P_0 \subseteq Q \subsetneq P} \mu(Q, P)\, \varphi(Q). \tag{29}$$

Each $Q \in \mathscr{I}(\mathbb{A})$ such that $P_0 \subseteq Q \subsetneq P$ can be expressed uniquely in the form $P_0 \vee x_I$, where $I \subseteq \{1, 2, ..., m\}$ and $x_I = \bigvee_{i \in I} x_i$. Recall that the join $\vee$ denotes convex hull. (See Section 1 for the definition of the meet $\wedge$ and join $\vee$ of polytopes in $\mathscr{I}(\mathbb{A})$.)

In this case, Proposition 1.1 implies that

$$\mu(Q, P) = \mu(P_0 \vee x_I, P) = (-1)^{m - |I|},$$

where $|I|$ denotes the size (cardinality) of the finite set I. We are now able to reformulate (29) as an inclusion-exclusion principle for polytope valuations φ with respect to the convex hull join $\vee$.

COROLLARY 8.3. *If $P \in \mathscr{I}(\mathbb{A})$ is not free and has m extreme points then*

$$\varphi(P) = \sum_{I \subsetneq \{1, 2, ..., m\}} (-1)^{m-|I|+1} \varphi(P_0 \vee x_I),$$

where P_0 and x_I are defined as above.

Similar relations exist for the $\wedge$ operation. For each extreme point x_i of P define

$$P_i = P_0 \vee x_1 \vee \cdots \vee \hat{x}_i \vee \cdots \vee x_m,$$

where $\hat{x}_i$ denotes *omission* of x_i from the convex hull join. Evidently $P = P_1 \vee \cdots \vee P_m$, while $P_0 = P_1 \wedge \cdots \wedge P_m$. In analogy to Corollary 8.3 we have the following inclusion–exclusion principle for polytope valuations with respect to the meet $\wedge$.

COROLLARY 8.4. *If $P \in \mathscr{I}(\mathbb{A})$ is not free then*

$$\varphi(P_0) = \sum_{I \subsetneq \{1, 2, ..., m\}} (-1)^{m-|I|+1} \varphi(P_{i_1} \wedge \cdots \wedge P_{i_{|I|}}), \tag{30}$$

where $I = \{i_1, i_2, ..., i_{|I|}\}$ and each P_i is defined as above.

Proof. From the identity (29) we obtain

$$\sum_{P_0 \subseteq Q \subseteq P} \mu(Q, P)\, \varphi(Q) = 0.$$

It follows that

$$\begin{aligned} \mu(P_0, P)\, \varphi(P_0) &= -\sum_{P_0 \subsetneq Q \subseteq P} \mu(Q, P)\, \varphi(Q) \\ &= \sum_{\varnothing \subsetneq J \subseteq \{1, 2, ..., m\}} (-1)^{m-|J|+1} \varphi(P_0 \vee x_J). \end{aligned} \tag{31}$$

For all $J \subseteq \{1, 2, ..., m\}$, let $I = \{i_1, i_2, ..., i_{|I|}\}$ denote the complement of J in $\{1, 2, ..., m\}$. We then have $P_0 \vee x_J = P_{i_1} \wedge \cdots \wedge P_{i_{|I|}}$ and $|I| + |J| = m$. Recall also that $\mu(P_0, P) = (-1)^m$. After making appropriate substitutions in (31) we obtain (30). ▮

9. INVARIANT POLYTOPE FUNCTIONALS

Let $\mathbb{Z}^n$ denote the set of all points in $\mathbb{R}^n$ having integer coordinates. We consider an application of the preceding results to the case of $\mathbb{A} = \mathbb{Z}^n$. Denote by $\mathscr{I}(\mathbb{Z}^n)$ the set of all *convex* polytopes having all of their vertices in $\mathbb{Z}^n$. Elements of $\mathscr{I}(\mathbb{Z}^n)$ will also be referred to as convex *integral polytopes.*

The group of integer translations of $\mathbb{R}^n$, also denoted $\mathbb{Z}^n$, acts on the lattice $\mathscr{I}(\mathbb{Z}^n)$. Moreover, the stabilizer of any given polytope $P \in \mathscr{I}(\mathbb{Z}^n)$ is just the trivial group, while the orbits of this $\mathbb{Z}^n$-action are congruence classes of integral polytopes under translation. Denote by $[P]$ the $\mathbb{Z}^n$-orbit of a polytope P.

The lattice $\mathscr{B}(\mathbb{Z}^n)$ inherits the action of $\mathbb{Z}^n$ on $\mathscr{I}(\mathbb{Z}^n)$ as follows. For $A \in \mathscr{B}(\mathbb{Z}^n)$ and $x \in \mathbb{Z}^n$, define

$$A + x = \{P + x \mid P \in A\}.$$

A function $f\colon \mathscr{I} \to \mathbb{R}$ is said to be *integer translation invariant* or $\mathbb{Z}^n$-*invariant* if $f(P+x) = f(P)$ for all x in $\mathbb{Z}^n$. Recall from Section 2 that such a function will induce an invariant valuation $\tilde{f}\colon \mathscr{B}(\mathbb{Z}^n) \to \mathbb{R}$.

The following is an important example of an invariant valuation on $\mathscr{B}(\mathbb{A})$. For a fixed orbit $[P]$, define

$$\alpha_P(A) = |A \cap [P]|,$$

for all $A \in \mathscr{B}(\mathbb{Z}^n)$. In other words, α_P counts the number of polytopes in the set A that are congruent to P under integer translations. Evidently α_P is $\mathbb{Z}^n$-invariant. The valuation α_P is induced by the ($\mathbb{Z}^n$-invariant) function on $\mathscr{I}(\mathbb{Z}^n)$ which takes the value 1 on polytopes $Q \in [P]$ and is otherwise zero. Note that α_P is a valuation on $\mathscr{B}(\mathbb{A})$, but is *not* in general a polytope valuation (on $\mathscr{P}^n$).

In analogy to Hadwiger's Characterization Theorem [15, 16, 20] for invariant valuations on compact convex sets in $\mathbb{R}^n$, we have the following basis theorem for translation invariant valuations on $\mathscr{B}(\mathbb{Z}^n)$.

THEOREM 9.1 (Basis Theorem). *Suppose $\tilde{f}\colon \mathscr{B}(\mathbb{Z}^n) \to \mathbb{R}$ is invariant under the action of $\mathbb{Z}^n$. For all $A \in \mathscr{B}(\mathbb{Z}^n)$,*

$$\tilde{f}(A) = \sum_{[Q]} f(\{Q\})\, \alpha_Q(A)$$

For other combinatorial analogues of Hadwiger's Theorem, see also [4, 17, 18, 20]. Note also that $\mathbb{R}$ can be replaced by any abelian group as the range of $\tilde{f}$.

Proof. For $A \in \mathscr{B}(\mathbb{Z}^n)$,

$$\tilde{f}(A) = \sum_{Q \in A} f(\{Q\}) = \sum_{[Q]} f(\{Q\})\, \alpha_Q(A),$$

where we collect terms over congruence classes $[Q]$. ∎

Once again, it is important to distinguish between polytope valuations (on $\mathscr{P}^n$) and the valuations on the Boolean algebra $\mathscr{B}(\mathbb{Z}^n)$, which are in one-to-one correspondence with all *functions* on convex polytopes in $\mathscr{I}(\mathbb{Z}^n)$. Theorem 4.1 now implies that $\mathbb{Z}^n$-invariant *polytope valuations* can be expressed as linear combinations of the functions α_P where each P is *free*.

COROLLARY 9.2. *Suppose φ is a $\mathbb{Z}^n$-invariant valuation on $\mathscr{P}^n$. For all $A \in \mathscr{B}(\mathbb{Z}^n)$,*

$$\varphi(A) = \sum_{\{[P]:\, P \text{ free}\}} \varphi(\{P\})\, \alpha_P(A),$$

Proof. Recall from Theorem 4.1 that $\varphi(\{P\}) = 0$ when P is not free. The corollary then immediately follows from Theorem 9.1. ∎

Theorem 4.1 asserts that $\varphi(\{P\}) = -\mu(\varnothing, P)\, \varphi^*(P)$. From Proposition 1.1 we then obtain

$$\varphi(A) = \sum_{\{[P]:\, P \text{ free}\}} (-1)^{\alpha_1(P)+1}\, \varphi^*(P)\, \alpha_P(A),$$

for all $A \in \mathscr{B}(\mathbb{Z}^n)$.

In the treatment above we could replace the translation group $\mathbb{Z}^n$ with the affine unimodular group $Aff(\mathbb{Z}^n)$ generated by all integer translations and integer linear transformations with determinant ± 1. In this case we would obtain basis theorems for affine unimodular invariant polytope functionals, summing over affine unimodular invariant congruence classes in $\mathscr{I}(\mathbb{Z}^n)$, in analogy to Theorem 9.1. For affine unimodular invariant *valuations* the summation would be over all affine unimodular invariant congruence classes of *free polytopes* (by Theorem 4.1), in analogy to Corollary 9.2.

Evidently similar basis theorems could be also derived for polytopes using hexagonal point lattices, symmetric tilings of spheres and hyperbolic spaces, and many other locally finite families of polytopes compatible with the symmetry group of an underlying space.

Basis theorems of the form of Theorem 9.1 (or its classical antecedents, such as Hadwiger's characterization theorem [15, 16, 20]) are often used to evaluate *kinematic formulas* for the expected value of an invariant valuation φ on a random intersection of polytopes (where one polytope is fixed and the other is moved at random by a symmetry of the space). Classical kinematic formulas play a prominent role in integral geometry [20, 36, 38]. Kinematic formulas in combinatorial theory are treated in detail in [17] (see also [18, 20]), and the techniques of [17] apply as well in the context of $\mathscr{I}(\mathbb{Z}^n)$ and other locally finite polytope families admitting the action of a symmetry group.

10. CLOSING REMARKS AND OPEN QUESTIONS

There remain a number of open questions regarding the locally finite lattice $\mathscr{P}(\mathbb{A})$ and the important special case of $\mathscr{P}(\mathbb{Z}^n)$.

First, it is well known that Macdonald's relation and the Dehn–Sommerville equations hold not only for convex polytopes, but for triangulated compact manifolds with boundary. In this article we showed that the identities of Theorem 4.2 hold for all polytopes in $\mathscr{P}(\mathbb{A})$ having a visible decomposition (see Theorem 4.5). The author conjectures that these identities also hold for all *non-singular* polytopes P that can be triangulated using the points $P \cap \mathbb{A}$ as vertices and such that P is a topological (or even Eulerian) manifold with boundary. Using combinatorial arguments and an induction on dimension one can prove this conjectured generalization of Theorem 4.2 in the case of $\mathbb{A} \subseteq \mathbb{R}^2$, but this particular approach will break down in higher dimension. Nonetheless, the author believes Theorem 4.2 will be seen to hold for non-singular polytopes in $\mathscr{P}(\mathbb{A})$ for any locally finite $\mathbb{A} \subseteq \mathbb{R}^n$.

A second question concerns the behavior of the free polytope enumerators with respect to dilation in the integer lattice. It is well known that, for $P \in \mathscr{P}(\mathbb{Z}^n)$, and positive integers k, the lattice point enumerator $\alpha_1(kP)$ is a polynomial:

$$\alpha_1(kP) = G_n(P)\, k^n + G_{n-1}(P)\, k^{n-1} + \cdots + G_1(P)\, k + \chi(P),$$

where $G_n(P) = \mathrm{Vol}(P)$, the n-dimensional volume of P, where $G_{n-1}(P)$ denotes the *relative surface area* of P, and where the remaining coefficients are other affine unimodular invariant valuations of P. This polynomial is known as the *Ehrhart polynomial* of P (see, for example, [13, 40]). One might ask, how do the functions $\alpha_i(kP)$ behave as functions of k, where P and i are fixed? Because the functionals α_i are *not* valuations for $i > 1$, their behavior under dilations, unions, and Minkowski sums remains mysterious.

Similarly, if $P \in \mathscr{P}(\mathbb{A})$ for some locally finite set $\mathbb{A}$, what can be said about how the functionals $\alpha_i(P)$ vary when $\mathbb{A}$ is replaced with a refinement $\mathbb{A}'$, such the barycentric subdivision with respect to some $\mathbb{A}$-triangulation of P?

The free polygon area formula, Theorem 7.2, should have interesting analogues for other locally finite families of points in $\mathbb{R}^2$ having appropriate symmetry. For example, one can show that if $\mathbb{A}$ is a hexagonal point lattice (such as would arise in the cross-section of a beehive), *free* polygons can have no more than 6 vertices, and an area formula analogous to Theorem 7.2 should not be too difficult to find.

In recent years much attention has been given to f-vectors, h-vectors, and the cd-index. Given a convex polytope P in $\mathbb{R}^n$, the f-vector of P is the vector

$$(f_0(P), f_1(P), ..., f_{n-1}(P)),$$

where $f_i(P)$ is the number of i-dimensional faces of P. The h-vector and cd-index are related expressions of face data for a polytope P. See, for example, [3, 11, 40, 41].

Consider instead a polytope $P \in \mathscr{I}(\mathbb{A})$. In place of the f-vector of P, one can consider the α-vector of P, namely, $\alpha = (\alpha_1(P), \alpha_2(P), ..., \alpha_m(P))$, where $\alpha_i(P)$ is the number of free polytopes $Q \subseteq P$ with $|Q \cap \mathbb{A}| = i$, and where m is the maximal number of vertices in a free polytope of $\mathscr{I}(\mathbb{A})$. Since the free polytope enumerators $\alpha_i(P)$ do not satisfy the Dehn–Sommerville equations (since the functionals α_i are not valuations for $i > 1$), one cannot immediately generalize the notions of h-vector and cd-index to the functionals α_i. On the other hand, the functionals α_i do satisfy a number of linear relations given by the free polytope inversion identity of Theorem 4.2, such as the free polytope Euler formula (5) and the interior point formula (25). It remains an open question how to find a complete set of linear relations satisfied by the α-vectors of polytopes in a locally finite collection $\mathscr{I}(\mathbb{A})$, and to determine which vectors are realizable as α-vectors of a polytope with respect to a given locally finite point set $\mathbb{A}$.

For convenience we have considered only real-valued polytope functionals in this article. One could consider instead functionals (and valuations) taking values in an arbitrary vector space V over a field K. In this instance, indicator functions for polytopes would be defined having images in the field K. It is also possible to replace the collection $\mathscr{P}^n$ (resp. $\mathscr{I}^n$) with the collection of all *rational* polytopes (resp. rational convex polytopes) in $\mathbb{R}^n$ and to consider rational-valued functionals on these polytopes. Groemer's integral and extension theorems for valuations on polytopes remain valid in these contexts [14], and the functional (and valuation) identities of Theorem 4.2 and its corollaries also remain valid, following the same arguments as those given above.

APPENDIX: NOMENCLATURE

$\mathbb{R}^n$	n-dimensional Euclidean space
$\mathbb{Z}^n$	Points of $\mathbb{R}^n$ having integer coordinates
$\mathbb{A}$	A locally finite subset of $\mathbb{R}^n$
$\mathscr{K}^n$	Set of all compact convex subsets of $\mathbb{R}^n$
$\mathscr{P}^n$	Set of all (possibly non-convex) polytopes in $\mathbb{R}^n$
$\mathscr{P}(\mathbb{A})$	Set of all polytopes $\mathscr{P}^n$ which can be triangulated with vertices in $\mathbb{A}$
$\mathscr{I}(\mathbb{A})$	Set of all convex polytopes having extreme points in $\mathbb{A}$
$\mathscr{B}(\mathbb{A})$	Power set of $\mathscr{I}(\mathbb{A})$
$\varnothing$	The empty set
$\bar{P}$	Set of all $Q \in \mathscr{I}(\mathbb{A})$ such that $Q \subseteq P$
$P \vee Q$	Convex hull of $P \cup Q$
$P \wedge Q$	Convex hull of $P \cap Q \cap \mathbb{A}$
$\alpha_i(A)$	Number of free polytopes in A having i vertices
χ	Euler characteristic
μ	Möbius function
$\tilde{f}$	Valuation on the Boolean algebra $\mathscr{B}(\mathbb{A})$ induced by a function f on $\mathscr{I}(\mathbb{A})$
ri P	Relative interior of P
∂P	Boundary of a non-singular polytope P
$\dim(P)$	Dimension of P
$\mathrm{Vis}(P)$	Visibility complex of P
$\Delta(P)$	Convex hull of $\mathrm{Vis}(P)$ (a simplex), also denoted Δ
π	Orthogonal projection of $\Delta(P)$ (or $\mathrm{Vis}(P)$) onto P
$\tilde{Q}$	The unique simplex in $\mathrm{Vis}(P)$ corresponding to a free polytope $Q \subseteq P$.
1_P	Indicator function of P
φ^*	Dual valuation of φ
$\mathrm{St}_P Q$	The (open) star of $\tilde{Q}$ in the complex $\mathrm{Vis}(P)$
$\mathrm{Ext}(P)$	Set of extreme points of the polytope P
$\mathrm{St}\, Q$	the (open) star of Q
$\overline{\mathrm{St}}\, Q$	The closed star of Q
$\mathrm{Lk}\, Q$	The link of Q
$\mathrm{Nb}_{\mathscr{C}}(Q)$	Set of closed cells in $\mathscr{C}$ that intersect Q
$f_k(\mathscr{T})$	Number of k-dimensional faces of a cell complex $\mathscr{T}$
$[P]$	Integer translative congruence class of P
$\alpha_P(A)$	Number of free polytopes in A that are congruent to P under $\mathbb{Z}^n$ translations
$\mathrm{vol}(P)$	Euclidean volume of P
$\mathrm{Area}(P)$	Euclidean area of P

REFERENCES

1. C. Ahrens, G. Gordon, and E. McMahon, Convexity and the beta invariant, preprint.
2. M. Bayer and L. Billera, Generalized Dehn–Sommerville relations for polytopes, spheres, and Eulerian partially ordered sets, *Invent. Math.* **79** (1985), 143–157.
3. M. Bayer and A. Klapper, A new index for polytopes, *Discrete Comput. Geom.* **6** (1991), 33–47.
4. U. Betke and M. Kneser, Zerlegungen und Bewertungen von Gitterpolytopen, *J. Reine Angew. Math.* **358** (1985), 202–208.
5. S. Brøndsted, "An Introduction to Convex Polytopes," Springer-Verlag, New York, 1983.
6. B. Chen, The geometric Dehn–Sommerville relations, *J. Combin. Theory Ser. A* **73** (1996), 142–151.
7. B. Chen, Weight functions, double reciprocity laws, and volume formulas for integral polyhedra, *Proc. Natl. Acad. Sci.*, to appear.
8. P. Edelman and R. Jamison, The theory of convex geometries, *Geom. Dedicata* **19** (1985), 247–270.
9. P. Edelman and V. Reiner, Counting the interior of a point configuration, preprint.
10. P. Edelman and V. Reiner, private communication.
11. R. Ehrenborg and M. Readdy, The **r**-cubical lattice and a generalization of the **cd**-index, *European J. Combin.* **17** (1996), 709–725.
12. P. Erdős, P. Gruber, and J. Hammer, "Lattice Points," Wiley, New York, 1989.
13. G. Ewald, "Combinatorial Convexity and Algebraic Geometry," Springer-Verlag, New York, 1996.
14. H. Groemer, On the extension of additive functionals on classes of convex sets, *Pacific J. Math.* **75** (1978), 397–410.
15. H. Hadwiger, "Vorlesungen über Inhalt, Oberfläche, und Isoperimetrie," Springer-Verlag, Berlin, 1957.
16. D. Klain, A short proof of Hadwiger's characterization theorem, *Mathematika* **42** (1995), 329–339.
17. D. Klain, Kinematic formulas for finite lattices, *Ann. Combinatorics* **1** (1997), 353–366.
18. D. Klain, Kinematic formulas for finite vector spaces, *Discrete Math.* **179** (1998), 121–132.
19. D. Klain, Interior point enumeration and the area of an integral polygon, *Discrete Math.*, to appear.
20. D. Klain and G.-C. Rota, "Introduction to Geometric Probability," Cambridge Univ. Press, New York, 1997.
21. V. Klee, A combinatorial analogue of Poincaré's duality theorem, *Canad. J. Math.* **16** (1964), 517–531.
22. J. Kung, Ed., "Gian-Carlo Rota on Combinatorics: Introductory Papers and Commentaries," Birkhäuser, Boston, 1995.
23. I. G. Macdonald, Polynomials associated with finite cell-complexes, *J. London Math. Soc.* **2** (4) (1971), 181–192.
24. P. McMullen, Non-linear angle-sum relations for polyhedral cones and polytopes, *Math. Proc. Cambridge Philos. Soc.* **78** (1975), 247–261.
25. P. McMullen, Valuations and Euler-type relations on certain classes of convex polytopes, *Proc. London Math. Soc.* **35** (1977), 113–135.
26. P. McMullen, The polytope algebra, *Adv. in Math.* **78** (1989), 76–130.
27. P. McMullen, Valuations and dissections, *in* "Handbook of Convex Geometry" (P. M. Gruber and J. M. Wills, Eds.), pp. 933–988, North-Holland, Amsterdam, 1993.
28. P. McMullen and R. Schneider, Valuations on convex bodies, *in* "Convexity and Its Applications" (P. M. Gruber and J. M. Wills, Eds.), pp. 170–247, Birkhäuser, Boston, 1983.

29. J. Munkres, “Elements of Algebraic Topology,” Benjamin/Cummings, Menlo Park, CA, 1984.
30. G. Pick, Geometrisches zur Zahlenlehre, *Naturwiss. Z. Lotus* (*Prag*) (1899), 311–319.
31. G.-C. Rota, On the foundations of combinatorial theory. I. Theory of Möbius functions, *Z. Wahrsch. 2* **368** (1964), 340–368.
32. G.-C. Rota, On the combinatorics of the Euler characteristic, *in* “Studies in Pure Mathematics (Papers Presented to Richard Rado),” pp. 221–233, Academic Press, London, 1971.
33. G.-C. Rota, The valuation ring of a distributive lattice, *in* “Proceedings of the University of Houston Lattice Theory Conference, Houston, TX, 1973,” pp. 574–628.
34. C.-H. Sah, “Hilbert’s Third Problem: Scissors Congruence,” Fearon Pitman, San Francisco, 1979.
35. G. T. Sallee, Polytopes, valuations, and the Euler relation, *Canad. J. Math.* **20** (1968), 1412–1424.
36. L. A. Santaló, “Integral Geometry and Geometric Probability,” Addison–Wesley, Reading, MA, 1976.
37. R. Schneider, Simple valuations on convex bodies, *Mathematika* **43** (1996), 32–39.
38. R. Schneider and J. A. Wieacker, Integral geometry, *in* “Handbook of Convex Geometry” (P. M. Gruber and J. M. Wills, Eds.), pp. 1349–1390, North-Holland, Amsterdam, 1993.
39. R. Stanley, Combinatorial reciprocity theorems, *Adv. Math.* **14** (1974), 194–253.
40. R. Stanley, “Enumerative Combinatorics,” Wadsworth & Brooks/Cole, Monterey, CA, 1986.
41. R. Stanley, Flag *f*-vectors and the *cd*-index, *Math. Z.* **216** (1994), 483–499.
42. R. Stanley, “Combinatorics and Commutative Algebra,” 2nd ed., Birkhäuser, Boston, 1996.
43. G. Ziegler, “Lectures on Polytopes,” Springer-Verlag, New York, 1995.